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Projectively Equivariant Quantization and Symbol calculus in dimension 1|2

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Abstract

The spaces of higher-order differential operators (in Dimension 1|2), which are modules over the stringy Lie superalgebra $\mathcal{K}(2)$, are isomorphic to the corresponding spaces of symbols as orthosymplectic modules in non resonant cases. Such an osp $(2|2)$-equivariant quantization, which has been given in second-order differential operators case, keeps existing and unique. We calculate its explicit formula that provides extension in particular order cases.

1 Introduction and the main results

Let $S^{1|2}$ be a supermanifold which is endowed with a projective structure (Susy-structure), see [12], in dimension 1|2 and $\mathcal{D}_{\lambda,\mu}\left(S^{1|2}\right)$ the space of differential operators on $S^{1|2}$ acting from the space of λ-densities to the space of μ-densities where λ and μ are real or complex numbers. The space $\mathcal{D}_{\lambda,\mu}\left(S^{1|2}\right)$ which is a module over the stringy superalgebra $\mathcal{K}(2)$, see [6], is naturally filtrated and has an other finer filtration by the contact order of differential operators. The space of symbols $\mathcal{S}\left(S^{1|2}\right)$, that is the graded module $\mathrm{gr}\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right)$, isn't isomorphic to the space $\mathcal{D}_{\lambda,\mu}\left(S^{1|2}\right)$ as $\mathcal{K}(2)$-module. Therefore, we have restricted the module structure on $\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right)$ to the orthosymplectic Lie superalgebra osp $(2|2)$ that is naturally embedded into $\mathcal{K}(2)$. We establish a canonical isomorphism between the space of differential operators on $S^{1|2}$ and the corresponding space of symbols. An explicit expression of projectively equivariant quantization map is given in case of second order differential operetors, see [12]. We extand calculus to symbols of higher order differential operators.

2 Geometry of the supercircle $S^{1|2}$

We have considered the supercircle $S^{1|2}$ described in [12] by its graded commutative algebra of complex-valued functions $C^\infty\left(S^{1|2}\right)$, consisting of the following elements :

$$f\left(x,\xi_1,\xi_2\right) = f_0(x) + \xi_1\, f_1(x) + \xi_2\, f_2(x) + \xi_1\xi_2\, f_{12}(x), \tag{2.1}$$

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where x is the Fourier image of the angle parameter on S^1, ξ_1, ξ_2 are odd Grassmann coordinates and $f_0, f_{12}, f_1, f_2 \in C^\infty(S)$ are functions with complex values. We have defined the parity function p by setting $p(x) = 0$ and $p(\xi_i) = 1$.

The *standard contact* structure on $S^{1|2}$, known as *Susy*-structure, is defined by the data of a linear distribution $\langle \overline{D}_1, \overline{D}_2 \rangle$ on $S^{1|2}$ generated by the odd vector fields :

$$\overline{D}_1 = \partial_{\xi_1} - \xi_1 \partial_x, \qquad \overline{D}_2 = \partial_{\xi_2} - \xi_2 \partial_x. \tag{2.2}$$

We would rather recall that every contact vector field can be expressed, for some function $f \in C^\infty\left(S^{1|2}\right)$, by

$$X_f = f\partial_x - (-1)^{p(f)} \frac{1}{2} \left(\overline{D}_1(f)\, \overline{D}_1 + \overline{D}_2(f)\, \overline{D}_2\right) \tag{2.3}$$

The *projective (conformal)* structure on the supercircle $S^{1|2}$, see [14], is defined by the action of Lie superalgebra $\mathrm{osp}(2|2)$. The *orthosymplectic* algebra $\mathrm{osp}(2|2)$ is spanned by the contact vector fields X_f with the contact Hamiltonians f which are elements of $\{1, \xi_1, \xi_2, x, \xi_1\xi_2, x\xi_1, x\xi_2, x^2\}$. The embedding of $\mathrm{osp}(2|2)$ into $\mathcal{K}(2)$ is given by (2.3). The subalgebra $\mathrm{Aff}(2|2)$ of $\mathrm{osp}(2|2)$, called the *Affine* Lie superalgebra, is spanned by the contact vector fields X_f with the contact Hamiltonians f which are elements of $\{1, \xi_1, \xi_2, x, \xi_1\xi_2\}$.

For any contact vector field, we have defined a family of differential operators of order one on $C^\infty\left(S^{1|2}\right)$:

$$L^\lambda_{X_f} := X_f + \lambda f', \tag{2.4}$$

where the parameter λ is an arbitrary complex number. Thus, we have obtained a family of $\mathcal{K}(2)$-modules on $C^\infty\left(S^{1|2}\right)$ noted by $\mathcal{F}_\lambda\left(S^{1|2}\right)$ which are called the spaces of *weighted densities* of weight λ.

3 Differential operators on the spaces of weighted densities

In this section, we have introduced the space of differential operators acting on the spaces of weighted densities. We have also presented the corresponding space of symbols on $S^{1|2}$. Those are detailed in [10, 7, 5, 2, 12].

For every integer or half-integer k, the space of differential operators of the form

$$A = \sum_{\ell + \frac{m}{2} + \frac{n}{2} \leq k} a_{\ell,m,n} \partial_x^\ell\, \overline{D}_1^m\, \overline{D}_2^n, \tag{3.1}$$

where $a_{\ell,m,n} \in C^\infty\left(S^{1|2}\right)$ and $m, n \leq 1$, has been noted by $\mathcal{D}^k_{\lambda\mu}\left(S^{1|2}\right)$. This above $\mathcal{K}(2)$-module space has a $\mathcal{K}(2)$-invariant *finer filtration* :

$$\mathcal{D}^0_{\lambda\mu}\left(S^{1|2}\right) \subset \mathcal{D}^{\frac{1}{2}}_{\lambda\mu}\left(S^{1|2}\right) \subset \mathcal{D}^1_{\lambda\mu}\left(S^{1|2}\right) \subset ... \subset \mathcal{D}^k_{\lambda\mu}\left(S^{1|2}\right) \subset \mathcal{D}^{k+\frac{1}{2}}_{\lambda\mu}\left(S^{1|2}\right) \subset \cdots \tag{3.2}$$

that has been considered in papers [12, 5].

We would rather remind that the orthosymplectic superalgebra osp $(2|2)$ has been acting on $\mathcal{D}^k_{\lambda\mu}\left(S^{1|2}\right)$. The action of contact field X_f of order one on a differential operator A, see formula (2.3), is given by

$$\pounds^{\lambda\mu}_{X_f}(A) = L^{\mu}_{X_f} \circ A - (-1)^{p(f)p(A)} A \circ L^{\lambda}_{X_f}. \tag{3.3}$$

3.1 Space of symbols of differential operators

The graded $\mathcal{K}(2)$-module, which is associated with the finer filtration (3.2) and called the *space of symbols* of differential operators, is defined by

$$\mathrm{gr}\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right) = \bigoplus_{i=0}^{\infty} \mathrm{gr}^{\frac{i}{2}}\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right), \tag{3.4}$$

where $\mathrm{gr}^k\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right) = \mathcal{D}^k_{\lambda\mu}\left(S^{1|2}\right)/\mathcal{D}^{k-\frac{1}{2}}_{\lambda\mu}\left(S^{1|2}\right)$ for every integer or half-integer k.

The image of any differential operator through the natural projection

$$\sigma_{pr} : \mathcal{D}^k_{\lambda\mu}\left(S^{1|2}\right) \to \mathrm{gr}^k\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right),$$

that is defined by the filtration (3.2), has been called the *principal symbol*.

Referring to [12], the stringy superalgebra $\mathcal{K}(2)$ keeps acting on the space of symbols.

Proposition 3.1. *If k is an integer, then*

$$\mathrm{gr}^k\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right) = \mathcal{F}_{\mu-\lambda-k} \bigoplus \mathcal{F}_{\mu-\lambda-k} \tag{3.5}$$

Proof. By definition (see formula (3.1)), a given operator $A \in \mathcal{D}^k_{\lambda\mu}\left(S^{1|2}\right)$ with integer k is of the form

$$A = F_1\,\partial_x^k + F_2\,\partial_x^{k-1}\overline{D}_1\overline{D}_2 + \cdots,$$

where $\cdots$ stand for lower order terms. The principal symbol of A is then encoded by the pair (F_1, F_2). From (3.3), we easily calculate the $\mathcal{K}(2)$-action on the principal symbol:

$$L_{X_f}(F_1, F_2) = \left(L^{\mu-\lambda-k}_{X_f}(F_1), L^{\mu-\lambda-k}_{X_f}(F_2).\right)$$

In other words, both F_1 and F_2 transform as $(\mu-\lambda-k)$-densities. $\square$

The situation is more complicated for half-integer k : the $\mathcal{K}(2)$-action has been given by

$$L_{X_f}(F_1, F_2) = (L^{\mu-\lambda-k}_{X_f}(F_1) - \frac{1}{2}\overline{D}_1\overline{D}_2(f)F_2,\ L^{\mu-\lambda-k}_{X_f}(F_2) + \frac{1}{2}\overline{D}_1\overline{D}_2(f)F_1). \tag{3.6}$$

Therefore, the spaces of symbols of half-integer contact order aren't isomorphic to the spaces of weighted densities.

Simplifying the notation as in [8, 3, 5, 12], the whole space of symbols $\mathrm{gr}\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right)$, depending only on $\mu-\lambda$, has been noted by $\mathcal{S}_{\mu-\lambda}\left(S^{1|2}\right)$, and the space of symbols of contact order k has been noted by $\mathcal{S}^k_{\mu-\lambda}\left(S^{1|2}\right)$.

A linear map, $Q : \mathcal{S}_{\mu-\lambda}\left(S^{1|2}\right) \to \mathcal{D}_{\lambda\mu}\left(S^{1|2}\right)$, is called a *quantization map* if it verifies bijectivety and preserves the principal symbol of every differential operator.

4 Projectively equivariant quantization on $S^{1|2}$

The main result of this paper is the existence and uniqueness of an $\mathrm{osp}(2|2)$-equivariant quantization map in Dimension $1|2$. We calculate its explicit formula.

For every m integer or half-integer, the space $\mathcal{D}^m_{\lambda\mu}\left(S^{1|2}\right)$ is isomorphic to the corresponding space of symbols as an $\mathrm{Aff}\,(2|2)$-module. We will show how to extend this isomorphism to that of the $\mathrm{osp}\,(2|2)$-modules.

4.1 The Divergence operators as Affine equivariant

Let us introduce new differential operators, which are called Divergence operators, on the space of symbols $\mathcal{S}_{\mu-\lambda}\left(S^{1|2}\right)$.

At first, we consider the case of differential operators of contact order k, where k is an integer. We have assumed that the symbols of differential operators are homogeneous and we have defined parity of the non vanished symbol (F_1, F_2) as $p(F) := p(F_1) = p(F_2)$.

4.1.1 The Divergence operators in case of integer contact order k

In this case, we define the Divergence as Affine equivariant differential operators on the space of symbols $\mathcal{S}_{\mu-\lambda}\left(S^{1|2}\right)$. In each component $\mathcal{S}^k_{\mu-\lambda}\left(S^{1|2}\right)$, we have

$$DIV^{2n+1}(F_1,F_2) = (-1)^{p(F)+1}\begin{pmatrix} \frac{k+2\lambda}{k}\partial_x^n\overline{D}_2(F_2) + \partial_x^n\overline{D}_1(F_1) \\ \partial_x^n\overline{D}_2(F_1) - \frac{k+2\lambda}{k}\partial_x^n\overline{D}_1(F_2) \end{pmatrix}, \tag{4.1}$$

$$DIV^{2n}(F_1,F_2) = \begin{pmatrix} \partial_x^n(F_1) - \frac{(k+2\lambda)n}{k(2(\mu-\lambda)+n-2k)}\partial_x^{n-1}\overline{D}_1\overline{D}_2(F_2) \\ \frac{(k+2\lambda)(k-n)}{k(k-n+2\lambda)}\partial_x^n(F_2) + \frac{n(k-n)}{(2(\mu-\lambda)+n-2k)(k-n+2\lambda)}\partial_x^{n-1}\overline{D}_1\overline{D}_2(F_1) \end{pmatrix} \tag{4.2}$$

and

$$\begin{aligned} div^{2k-(2n+1)} &= \left(\ \partial_x^{k-n-1}\overline{D}_1,\quad \partial_x^{k-n-1}\overline{D}_2\ \right), \\ div^{2k-(2n)} &= \left(\ \partial_x^{k-n},\quad \partial_x^{k-n-1}\overline{D}_1\overline{D}_2\ \right). \end{aligned}$$

Lemma 4.1. *The Divergence operators (4.1) and (4.2) commute with the* $\mathrm{Aff}\,(2|2)$*-action.*

Proof. This is a direct consequence of projectively equivariant symbol calculus.

We are looking for the symbols:

$$DIV^{2n+1}(F_1,F_2) = \begin{pmatrix} c_1\partial_x^n\overline{D}_2(F_2) + c_2\partial_x^n\overline{D}_1(F_1) \\ c_3\partial_x^n\overline{D}_2(F_1) + c_4\partial_x^n\overline{D}_1(F_2) \end{pmatrix}$$

and

$$DIV^{2n}(F_1,F_2) = \begin{pmatrix} c_5\partial_x^n(F_1) + c_6\partial_x^{n-1}\overline{D}_1\overline{D}_2(F_2) \\ c_7\partial_x^n(F_2) + c_8\partial_x^{n-1}\overline{D}_1\overline{D}_2(F_1) \end{pmatrix}$$

where $c_i(1 \le i \le 8)$ are arbitrary constants. From the commutation relation $\left[L_{X_f}, DIV\right]$ for $f \in \mathrm{Aff}\,(2|2)$ we easily get the $\mathrm{Aff}\,(2|2)$-equivariance if Divergence operators . $\square$

4.1.2 The Divergence operators in case of half-integer contact order $k+\frac{1}{2}$

In this case, we also define the Divergence as Affine equivariant differential operators on the space of symbols $\mathcal{S}_{\mu-\lambda}\left(S^{1|2}\right)$. In each component $\mathcal{S}_{\mu-\lambda}^{k+\frac{1}{2}}\left(S^{1|2}\right)$ we have

$$DIV^{2n+1}\left(F_1,F_2\right)=(-1)^{p(F)}\frac{2\left(\mu-\lambda\right)-\left(2k+1\right)}{2\left(\mu-\lambda\right)-2k}\begin{pmatrix}\partial_x^n\overline{D}_2\left(F_2\right)+\partial_x^n\overline{D}_1\left(F_1\right)\\ \frac{k-n}{k-n+2\lambda}\left(\partial_x^n\overline{D}_2\left(F_1\right)-\partial_x^n\overline{D}_1\left(F_2\right)\right)\end{pmatrix}, \tag{4.3}$$

$$DIV^{2n}\left(F_1,F_2\right)=\frac{2\left(\mu-\lambda\right)-\left(2k+1\right)}{2\left(\mu-\lambda\right)-2k}\begin{pmatrix}\frac{2(\mu-\lambda)+n-2k}{2(\mu-\lambda)+n-(2k+1)}\partial_x^n\left(F_1\right)-\frac{n}{2(\mu-\lambda)+n-(2k+1)}\partial_x^{n-1}\overline{D}_1\overline{D}_2\left(F_2\right)\\ \frac{2(\mu-\lambda)+n-2k}{2(\mu-\lambda)+n-(2k+1)}\partial_x^n\left(F_2\right)+\frac{n}{2(\mu-\lambda)+n-(2k+1)}\partial_x^{n-1}\overline{D}_1\overline{D}_2\left(F_1\right)\end{pmatrix}, \tag{4.4}$$

and

$$\begin{aligned} div^{2k+1-(2n+1)} &= \left(\ \partial_x^{k-n},\quad \partial_x^{k-n-1}\overline{D}_1\overline{D}_2\ \right),\\ div^{2k+1-(2n)} &= \left(\ \partial_x^{k-n}\overline{D}_1,\quad \partial_x^{k-n}\overline{D}_2\ \right).\end{aligned}$$

Lemma 4.2. *The Divergence operators (4.3) and (4.4) commute with the action of Affine Lie superalgebra.*

Proof. Straightforward calculus. □

4.2 Staitement of the main result

Let us give the explicit formula of the projectively equivariant quantization map. We will give the proof in the next section.

Theorem 4.3. *The unique* $\mathrm{osp}\left(2|2\right)$*-equivariant quantization map associates the following differential operator with a symbol* $\left(F_1,F_2\right)\in\mathcal{S}_{\mu-\lambda}^{k}\left(S^{1|2}\right)$ *where* k *is (even or odd) integer :*

$$Q\left(F_1,F_2\right)=\sum_{n=0}^{k}\frac{\begin{pmatrix}\left[\frac{k}{2}\right]\\ \left[\frac{2n+1+(-1)^k}{4}\right]\end{pmatrix}\begin{pmatrix}\left[\frac{k-1}{2}\right]+2\lambda\\ \left[\frac{2n+1-(-1)^k}{4}\right]\end{pmatrix}}{\begin{pmatrix}k-2\left(\mu-\lambda\right)\\ \left[\frac{n+1}{2}\right]\end{pmatrix}}DIV^n\left(F_1,F_2\right)div^{k-n} \tag{4.5}$$

provided $\mu-\lambda\neq0,\frac{1}{2},1,\frac{3}{2},2...$*where* DIV *and* div *are defined in each particular case of even or odd contact order : (4.1), (4.2), (4.3) and (4.4) and the coefficients are* $\begin{pmatrix}n\\ m\end{pmatrix}=\frac{n(n-1)..(n-m+1)}{m!}$.

Remark 4.4. *This theorem keeps being achieved in the case of Dimension* $1|1$*, see [5]; the divergence operators* DIV *and* div *are given by* $\overline{D}$.

4.3 Proof of theorem in case of k-differential operators

Proof. Let us first consider the case of k-differential operators where k is integer. The quantization map 4.5 is, indeed, osp$(2|2)$-equivariant. Now, we are considering a differentiable linear map $Q : \mathcal{S}^k_{\mu-\lambda}\left(S^{1|2}\right) \to \mathcal{D}^k_{\lambda\mu}\left(S^{1|2}\right)$ for $k \geq 1$, preserving the principal symbol. Such a map is of the form :

$$\begin{aligned} Q(F_1, F_2) &= F_1\partial_x^k + F_2\partial_x^{k-1}\overline{D}_1\overline{D}_2 + ... \\ &+\widetilde{Q}_1^{(2\ell)}(F_1) + \widetilde{Q}_1^{(2\ell+1)}(F_1) \\ &+\widetilde{Q}_2^{(2\ell)}(F_2) + \widetilde{Q}_2^{(2\ell+1)}(F_2) \\ &.. + \left(C_{2k,1}\partial_x^k(F_1) + C_{2k,2}\partial_x^{k-1}\overline{D}_1\overline{D}_2(F_2)\right) \end{aligned}$$

where $\widetilde{Q}_1^{(m)}$ and $\widetilde{Q}_2^{(m)}$ are differential operators with coefficients in $\mathcal{F}_{\mu-\lambda}\left(S^{1|2}\right)$, see (3.1).

We obtain the following :

a) This map commutes with the action of the vector fields $D_1, D_2 \in$ osp$(2|2)$, where $D_i = \partial_{\xi_i} + \xi_i\partial_x$, if and only if the differential operators $\widetilde{Q}_1^{(m)}$ and $\widetilde{Q}_2^{(m)}$ are with constant coefficients.

b) This map commutes with the linear vector fields $X_{\xi_1}, X_{\xi_2}, X_x$ if and only if the differential operators $\widetilde{Q}_1^{(m)}$ and $\widetilde{Q}_2^{(m)}$ are of contact order $\frac{m}{2}$ in addition to the form

$$\begin{cases} \widetilde{Q}_1^{(2\ell+1)}(F_1) = C_{2\ell+1,1}\partial_x^\ell\overline{D}_1(F_1)\partial_x^{k-\ell-1}\overline{D}_1 + C_{2\ell+1,3}\partial_x^\ell\overline{D}_2(F_1)\partial_x^{k-\ell-1}\overline{D}_2 \\ \widetilde{Q}_1^{(2\ell)}(F_1) = C_{2\ell,1}\partial_x^\ell(F_1)\partial_x^{k-\ell} + C_{2\ell,3}\partial_x^{\ell-1}\overline{D}_1\overline{D}_2(F_1)\partial_x^{k-\ell-1}\overline{D}_1\overline{D}_2 \\ \widetilde{Q}_2^{(2\ell+1)}(F_2) = C_{2\ell+1,2}\partial_x^\ell\overline{D}_2(F_2)\partial_x^{k-\ell-1}\overline{D}_1 + C_{2\ell+1,4}\partial_x^\ell\overline{D}_1(F_2)\partial_x^{k-\ell-1}\overline{D}_2 \\ \widetilde{Q}_2^{(2\ell)}(F_2) = C_{2\ell,2}\partial_x^{\ell-1}\overline{D}_1\overline{D}_2(F_2)\partial_x^{k-\ell} + C_{2\ell,4}\partial_x^\ell(F_2)\partial_x^{k-\ell-1}\overline{D}_1\overline{D}_2 \end{cases}$$

where the coefficients $C_{m,i}(i = 1, 2, 3, 4)$ are arbitrary constants.

Note that the vector field $X_{x\xi_i}$ is the commutation relation $[X_{\xi_i}, X_{x^2}], i = 1, 2$, so it is sufficient to impose the equivariance with respect to the vector field X_{x^2} to meet the whole condition of osp$(2|2)$-equivariance.

d) The above quantization map commutes with the action of X_{x^2} if and only if any the coefficients $C_{m,i}$ verify the following conditions :

$$\begin{cases} \ell(\ell-1+2(\mu-\lambda-k))C_{2\ell,1} = -(k-\ell+1)(k-\ell+2\lambda)C_{2\ell-2,1} \\ (\ell+1)(\ell+2(\mu-\lambda-k))C_{2\ell,2} = (k-\ell+2\lambda)\begin{pmatrix} (-1)^{p(F)}(C_{2\ell-1,2}-C_{2\ell-1,4}) \\ -(k-\ell+1)C_{2\ell-2,2} \end{pmatrix} \\ (\ell+1)(\ell+2(\mu-\lambda-k))C_{2\ell,3} = -(k-\ell)\begin{pmatrix} (-1)^{p(F)}(C_{2\ell-1,1}+C_{2\ell-1,3}) \\ +(k-\ell+2\lambda+1)C_{2\ell-2,3} \end{pmatrix} \\ \ell(\ell-1+2(\mu-\lambda-k))C_{2\ell,4} = -(k-\ell)(k-\ell+2\lambda+1)C_{2\ell-2,4} \\ (\ell+1)(\ell+2(\mu-\lambda-k))C_{2\ell+1,1} = (k-\ell)\left((-1)^{p(F)}C_{2\ell,1}-(k-\ell+2\lambda)C_{2\ell-1,1}\right) \\ (\ell+1)(\ell+2(\mu-\lambda-k))C_{2\ell+1,2} = (k-\ell+2\lambda)\left((-1)^{p(F)}C_{2\ell,4}-(k-\ell)C_{2\ell-1,2}\right) \\ (\ell+1)(\ell+2(\mu-\lambda-k))C_{2\ell+1,3} = (k-\ell)\left((-1)^{p(F)}C_{2\ell,1}-(k-\ell+2\lambda)C_{2\ell-1,3}\right) \\ (\ell+1)(\ell+2(\mu-\lambda-k))C_{2\ell+1,4} = -(k-\ell+2\lambda)\left((-1)^{p(F)}C_{2\ell,4}+(k-\ell)C_{2\ell-1,4}\right) \end{cases}$$

If $\mu-\lambda \neq 0, \frac{1}{2}, 1, \frac{3}{2}, 2...$, this system has been solved and the solutions are the following:

$$\begin{cases} C_{2\ell,2} = \dfrac{\begin{pmatrix} k-1 \\ \ell-1 \end{pmatrix}\begin{pmatrix} k+2\lambda \\ \ell+1 \end{pmatrix}}{\begin{pmatrix} -2(\mu-\lambda-k) \\ \ell+1 \end{pmatrix}} \\ C_{2\ell,3} = -\dfrac{\begin{pmatrix} k \\ \ell+1 \end{pmatrix}\begin{pmatrix} k+2\lambda-1 \\ \ell-1 \end{pmatrix}}{\begin{pmatrix} -2(\mu-\lambda-k) \\ \ell+1 \end{pmatrix}} \end{cases} \text{ and } \begin{cases} C_{2\ell,1} = \frac{(k+2\lambda-\ell)(2(\mu-\lambda-k)+\ell)}{\ell(k-\ell)} C_{2\ell,3}, \\ C_{2\ell,4} = -\frac{(k-\ell)(2(\mu-\lambda-k)+\ell)}{\ell(k-\ell+2\lambda)} C_{2l,2}, \\ C_{2\ell+1,1} = (-1)^{p(F)} \frac{(k+2\lambda-\ell)}{\ell} C_{2\ell,3}, \\ C_{2\ell+1,2} = -(-1)^{p(F)} \frac{(k-\ell)}{\ell} C_{2\ell,2}, \\ C_{2\ell+1,3} = (-1)^{p(F)} \frac{(k+2\lambda-\ell)}{\ell} C_{2\ell,3} \\ C_{2\ell+1,4} = (-1)^{p(F)} \frac{(k-\ell)}{\ell} C_{2\ell,2} \end{cases}.$$

That allows us to obtain the formula (4.5). □

4.4 Proof of theorem in case of $\left(k+\frac{1}{2}\right)$-differential operators

Proof. In the case of $\left(k+\frac{1}{2}\right)$-differential operators where k is integer, we get an Aff $(2|2)$-equivariant quantization map by a straightforward calculation which is given by

$$\begin{aligned} Q(F_1, F_2) &= F_1 \partial_x^k \overline{D}_1 + F_2 \partial_x^k \overline{D}_2 + ... \\ &\quad + \widetilde{Q}_1^{(2\ell)}(F_1) + \widetilde{Q}_1^{(2\ell+1)}(F_1) \\ &\quad + \widetilde{Q}_2^{(2\ell)}(F_2) + \widetilde{Q}_2^{(2\ell+1)}(F_2) \\ &\quad .. + \left(C_{2k+1,1} \partial_x^k \overline{D}_1(F_1) + C_{2k+1,2} \partial_x^k \overline{D}_2(F_2)\right) \end{aligned}$$

where the $\frac{m}{2}$-differential operators $\widetilde{Q}_1^{(m)}$ and $\widetilde{Q}_2^{(m)}$ have the form :

$$\begin{cases} \widetilde{Q}_1^{(2\ell)}(F_1) = C_{2\ell,1} \partial_x^\ell(F_1) \partial_x^{k-\ell} \overline{D}_1 + C_{2\ell,3} \partial_x^{\ell-1} \overline{D}_1 \overline{D}_2(F_1) \partial_x^{k-\ell} \overline{D}_2 \\ \widetilde{Q}_1^{(2\ell+1)}(F_1) = C_{2\ell+1,1} \partial_x^\ell \overline{D}_1(F_1) \partial_x^{k-\ell} + C_{2\ell+1,3} \partial_x^\ell \overline{D}_2(F_1) \partial_x^{k-\ell-1} \overline{D}_1 \overline{D}_2 \\ \widetilde{Q}_2^{(2\ell)}(F_2) = C_{2\ell,2} \partial_x^{\ell-1} \overline{D}_1 \overline{D}_2(F_2) \partial_x^{k-\ell} \overline{D}_1 + C_{2\ell,4} \partial_x^\ell(F_2) \partial_x^{k-\ell} \overline{D}_2 \\ \widetilde{Q}_2^{(2\ell+1)}(F_2) = C_{2\ell+1,2} \partial_x^\ell \overline{D}_2(F_2) \partial_x^{k-\ell} + C_{2\ell+1,4} \partial_x^\ell \overline{D}_1(F_2) \partial_x^{k-\ell-1} \overline{D}_1 \overline{D}_2 \end{cases}$$

The above quantization map commutes with the action of X_{x^2} if and only if the coefficients $C_{m,j} (j = 1, 2, 3, 4)$ verify the following system of linear equations :

$$
\left\{
\begin{array}{lcl}
\begin{pmatrix} C_{2\ell,1} \\ -(\ell+1)\left(\ell+2\begin{pmatrix}\mu-\lambda\\ -k-\frac{1}{2}\end{pmatrix}\right)C_{2\ell,2}\end{pmatrix} & = & \begin{pmatrix}(k-\ell+1)(k-\ell+2\lambda+1)\,C_{2\ell-2,2}\\ +(-1)^{p(F)}(k-\ell+1)\,C_{2\ell-1,2}\\ +C_{2\ell,4}\\ -(-1)^{p(F)}(k-\ell+2\lambda+1)\,C_{2\ell-1,4}\end{pmatrix}\\
\begin{pmatrix}(\ell+1)\left(\ell+2\begin{pmatrix}\mu-\lambda\\ -k-\frac{1}{2}\end{pmatrix}\right)C_{2\ell,3}\\ +C_{2\ell,4}\end{pmatrix} & = & \begin{pmatrix} C_{2l,1}\\ +(-1)^{p(F)}(k-\ell+1)\,C_{2\ell-1,1}\\ -(k-\ell+1)(k-\ell+2\lambda+1)\,C_{2\ell-2,3}\\ +(-1)^{p(F)}(k-\ell+2\lambda+1)\,C_{2\ell-1,3}\end{pmatrix}\\
\begin{pmatrix}\ell\left(\ell-1+2\left(\mu-\lambda-k-\frac{1}{2}\right)\right)C_{2\ell,1}\\ -C_{2\ell,2}\end{pmatrix} & = & -(k-\ell+1)(k-\ell+2\lambda+1)\,C_{2\ell-2,1}\\
\begin{pmatrix}C_{2\ell,3}\\ +\ell\left(\ell-1+2\left(\mu-\lambda-k-\frac{1}{2}\right)\right)C_{2\ell,4}\end{pmatrix} & = & -(k-\ell+1)(k-\ell+2\lambda+1)\,C_{2\ell-2,4}\\
\begin{pmatrix}C_{2\ell+1,1}\\ +(\ell+1)\left(\ell+2\begin{pmatrix}\mu-\lambda\\ -k-\frac{1}{2}\end{pmatrix}\right)C_{2\ell+1,2}\end{pmatrix} & = & -(k-\ell+2\lambda)\begin{pmatrix}(-1)^{p(F)}\,C_{2\ell,4}\\ +(k-\ell+1)\,C_{2\ell-1,2}\end{pmatrix}\\
\begin{pmatrix}C_{2\ell+1,2}\\ +(\ell+1)\left(\ell+2\begin{pmatrix}\mu-\lambda\\ -k-\frac{1}{2}\end{pmatrix}\right)C_{2\ell+1,1}\end{pmatrix} & = & -(k-\ell+2\lambda)\begin{pmatrix}(-1)^{p(F)}\,C_{2\ell,1}\\ +(k-\ell+1)\,C_{2\ell-1,1}\end{pmatrix}\\
\begin{pmatrix}C_{2\ell+1,3}\\ -(\ell+1)\left(\ell+2\begin{pmatrix}\mu-\lambda\\ -k-\frac{1}{2}\end{pmatrix}\right)C_{2\ell+1,4}\end{pmatrix} & = & (k-\ell)\begin{pmatrix}(k-\ell+2\lambda+1)\,C_{2\ell-1,4}\\ -(-1)^{p(F)}\,C_{2\ell,4}\end{pmatrix}\\
\begin{pmatrix}C_{2\ell+1,4}\\ -(\ell+1)\left(\ell+2\begin{pmatrix}\mu-\lambda\\ -k-\frac{1}{2}\end{pmatrix}\right)C_{2\ell+1,3}\end{pmatrix} & = & (k-\ell)\begin{pmatrix}(-1)^{p(F)}\,C_{2\ell,1}\\ +(k-\ell+2\lambda+1)\,C_{2\ell-1,3}\end{pmatrix}
\end{array}
\right.
$$

By solving this system, we obtain the formula (4.5). □

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